NsG-220-62

UNIVERSITY OF MARYLAND

THE INSTITUTE FOR FLUID DYNAMICS

and

APPLIED MATHEMATICS

Technical Note BN-351 March, 1964

A GENERALIZED PERTURBATION EXPANSION FOR THE KLEIN-GORDON EQUATION WITH A SMALL NONLINEARITY*

by

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* This work was supported in part by the National Aeronautics and Space Administration, under Contract Ns G 220-62.

ABSTRACT

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A previously-given method for deriving uniformly valid perturbation series for the Klein-Gordon equation with a "small" nonlinear term is generalized to include situations in which the lowest order solution is not restricted to be a monochromatic wave. Author

I. INTRODUCTION

It is the purpose of this paper to present a general method for developing well-behaved perturbation expansions in ϵ for the Klein-Gordon equation with a small nonlinear term:

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} + \lambda^2\right) f\,(x,t) \quad = \quad \epsilon F\ \left(f, \frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}\right) \quad (1a)$$

where F is some arbitrary function of f, $\partial f/\partial t$, and $\partial f/\partial x$. Both f and F are real. For $\varepsilon=0$, the solution to (1a) is easily given as a Fourier series or integral; however, a straightforward attempt to expand in powers of ε about the $\varepsilon = 0$ solution leads to secular (i.e., t or x proportional) terms in the corrections to f, making more refined methods necessary. Such an expansion for Eq. (1a) has recently been given[1], adapting the Krylov-Bogolyubov-Mitropolskii techniques of nonlinear mechanics [2,3]. However, the treatment of reference 1 (which was designed for a specific plasma problem) suffers from one rather severe limitation on the zeroth order solution: only a monochromatic wave led to manageable equations in the higher orders.

Here, this limitation is removed, and the treatment of reference 1 is generalized to include all situations in which the zeroth order is expressible as a Fourier series which is summed over a discrete spectrum of frequencies. The restriction to a discrete spectrum does not eliminate any interesting phenomena; the continuous spectrum case actually appears <u>less</u> pathological than the discrete one.

It turns out to be easier to work in the characteristic coordinates rather than x and t. Therefore, in Sec. II, the method is developed entirely in these coordinates. In Sec. III, the role of boundary conditions is discussed, and two simple examples are treated in Sec. IV.

In the interests of simplicity, we go only to $O(\varepsilon)$ in this paper, though nothing conceptually new is involved in going to higher order. Also for simplicity, we confine ourselves to very simple forms for F, for the F's which occur in problems of genuine physical interest seem always to generate so much algebra that an understanding of the method becomes unnecessarily difficult.

II. THE METHOD

The substitutions $\xi = t + x/c$, $\eta = t - x/c$, $\lambda^2 = 4\Lambda^2$, $F = 4\mathcal{F}$, reduce (1a) to the form

$$\left\{ \frac{\partial^2}{\partial\xi\partial\eta} + \Lambda^2 \right\} f \quad (\xi,\eta) = \varepsilon \mathcal{F} \left(f, \frac{\partial f}{\partial\xi}, \frac{\partial f}{\partial\eta}\right) \quad . \tag{1b}$$

We seek a solution to (1b) of the form

$$f = \sum_{K,L} a\ (K,L)\ e^{i\psi\ (K,L)} + \varepsilon u_1\ (a,\psi) + \varepsilon^2 u_2\ (a,\psi) + \ldots, \tag{2}$$

where "a" and " ψ " stand symbolically for all the amplitudes $a(K,L)$ and phases $\psi(K,L)$.

We must take some pains to identify the labels K,L; to do this requires several steps.

(i) We introduce an enumerable sequence of two-component basic vectors, (k_j,ℓ_j), the components of which satisfy

$$k_j\ell_j = \Lambda^2 \qquad \text{all } j.$$

The allowed values of k_j or ℓ_j will always be determined by boundary conditions, in a way that need not concern us yet.

(ii) The notation (K,L) means the derived vector

$$(K,L) = n_1 (k_1,\ell_1) + n_2 (k_2,\ell_2) + \dots + n_N (k_N,\ell_N),$$

where n_1, n_2, ..., n_N are any collection of integers, positive, negative, or zero. Clearly, the basic vectors are also derived vectors.

(iii) If

$$(K,L) = \textstyle\sum_j n_j (k_j,\ell_j)$$

and

$$(K',L') = \textstyle\sum_j n_j' (k_j,\ell_j),$$

then (K,L) is regarded as the same vector as (K′, L′) if and only if

$$n_j = n_j' , \qquad \text{all } j.$$

(iv) With each basic vector (k_j, ℓ_j) there is associated a basic phase $\psi(k_j, \ell_j)$.

(v) For the derived vector (K,L), the phase ψ (K,L) is defined by

$$\psi(K,L) = \psi(\textstyle\sum_j n_j k_j, \sum_j n_j \ell_j) = \sum_j n_j \psi(k_j, \ell_j).$$

(vi) For the basic phases $\psi(k_j, \ell_j)$, there is assumed to exist an expansion of the form

$$\frac{\partial\psi(k_j,\ell_j)}{\partial\xi} = k_j + \varepsilon C_{k_j \ell_j}(a) + \ldots \quad ,$$

$$\frac{\partial\psi(k_j,\ell_j)}{\partial\eta} = \ell_j + \varepsilon D_{k_j \ell_j}(a) + \ldots \quad , \tag{3}$$

where the coefficients $C_{k_j \ell_j}$, $D_{k_j \ell_j}$ depend only upon the amplitudes, but remain otherwise unspecified, as yet. The "..." means higher powers of ε.

These statements specify the kind of quantity over which the summations $\sum_{K,L}$ are to be carried out. Clearly, the ψ (K,L) also possess expansions of a form similar to (3). By the symbol k, we shall mean the numerical value of $n_1 k_1 + n_2 k_2 + \ldots + n_N k_N$, and by ℓ, the numerical value of $n_1 \ell_1 + n_2 \ell_2 + \ldots + n_N \ell_N$. Thus, (K,L) is a basic vector if and only if $k\ell = \Lambda^2$, but different basic vectors may have the same value of k and ℓ.

To finish specifying the sort of solution (2) that we are seeking, we assume for the a(K,L) expansions of the form

$$\frac{\partial a\,(K,L)}{\partial \xi} = \varepsilon A_{KL}\,(a) + \;.\;.\;.$$
$$\frac{\partial a\,(K,L)}{\partial \eta} = \varepsilon B_{KL}\,(a) + \;.\;.\;. \qquad (4)$$

and note that (v) and (vi) imply that for the derived phases,

$$\frac{\partial \psi\,(K,L)}{\partial \xi} = \sum_j n_j k_j + \varepsilon \sum_j n_j C_{k_j \ell_j}(a) + \ldots$$
$$\frac{\partial \psi\,(K,L)}{\partial \eta} = \sum_j n_j \ell_j + \varepsilon \sum_j n_j D_{k_j \ell_j}(a) + \ldots \qquad (5)$$

It is no loss of generality to assume that a (K,L), ψ (K,L) are real, and that

$$a\,(K,L) = a\,(-K,\ -L),$$
$$A_{KL}\,(a) = A_{-K,-L}\,(a),$$
$$B_{KL}\,(a) = B_{-K,-L}\,(a),$$
$$C_{k_j \ell_j}(a) = -C_{-k_j,\,-\ell_j}(a), \qquad (6)$$
$$D_{k_j \ell_j}(a) = -D_{-k_j,\,-\ell_j}(a),$$
$$\psi(K,L) = -\psi(-K,\ -L).$$

For the solution to be consistent for $\varepsilon = 0$, we require that

$$a\,(K,L) = O\,(\varepsilon), \qquad k\ell \neq \Lambda^2, \tag{7}$$

so that if (2) is written as

$$f = f^{(o)} + \varepsilon f^{(1)} + \ldots, \tag{8}$$

then $f^{(o)}$ may be unambiguously written as

$$f^{(o)} = \sum_{\substack{K,L \\ k\ell=\Lambda^2}} a\,(K,L)\, e^{i\psi(K,L)} \quad .$$

With this set of notations, and for any $\mathcal{F}$ which is expandable about zero in f, $\partial f/\partial\xi$, $\partial f/\partial\eta$, we may always write

$$\mathcal{F}(f^{(o)}, \partial f^{(o)}/\partial\xi, \partial f^{(o)}/\partial\eta) = \sum_{K,L} F_{KL}\,(a)\, e^{i\psi(K,L)} \tag{9}$$

where the $F_{KL}\,(a)$ are <u>known</u> functions of the amplitudes which satisfy, for $\mathcal{F}$ real,

$$F_{KL}\,(a) = F^*_{-K,\,-L}\,(a) \quad . \tag{10}$$

In practice, $\mathcal{F}$ will always be some polynomial in f, $\partial f/\partial\xi$, and $\partial f/\partial\eta$.

Using Eqs. (2) through (9) in Eq. (1b), we get, upon noting that the zeroth order coefficient vanishes identically and equating coefficients of ε:

$$\left(\frac{\partial^2}{\partial\xi\partial\eta} + \Lambda^2\right) u_1 (a,\psi) + \sum_{\substack{K,L \\ k\ell \neq \Lambda^2}} (-k\ell + \Lambda^2)\, a\,(K,L)\, e^{i\psi(K,L)}$$

$$+ \sum_{k_j,\ell_j} e^{i\psi(k_j,\ell_j)} \{ i(k_j B_{k_j\ell_j} + \ell_j A_{k_j\ell_j})$$

$$- a\,(k_j,\ell_j)\,(k_j D_{k_j\ell_j} + \ell_j C_{k_j\ell_j})\}$$

$$= \sum_{K,L} F_{KL}(a)\, e^{i\psi(K,L)}$$

If we define

$$v_1 = u_1 + \sum_{\substack{K,L \\ k\ell \neq \Lambda^2}} a\,(K,L)\, e^{i\psi(K,L)},$$

then (11) becomes a differential equation for v_1:

$$\left(\frac{\partial^2}{\partial\xi\partial\eta} + \Lambda^2\right) v_1 = \sum_{K,L} F_{KL}(a)\, e^{i\psi(K,L)}$$

$$- \sum_{k_j,\ell_j} e^{i\psi(k_j,\ell_j)} \{ i\,(k_j B_{k_j\ell_j} + \ell_j A_{k_j\ell_j}) \tag{12}$$

$$- a\,(k_j,\ell_j)\,(k_j D_{k_j\ell_j} + \ell_j C_{k_j\ell_j})\}$$

Equation (12) has a secularity free solution for v_1 of the form

$$v_1 = \sum_{K,L} v^{(1)}(K,L)\, e^{\,i\psi(K,L)}$$

if and only if

$$i\,(k_j B_{k_j \ell_j} + \ell_j A_{k_j \ell_j}) - a_{k_j \ell_j}\,(k_j D_{k_j \ell_j} + \ell_j C_{k_j \ell_j}) \qquad (13)$$

$$= F_{k_j \ell_j}(a)$$

for all (k_j, ℓ_j). Note that nowhere have we been required to commit ourselves as to the values of $A_{k_j \ell_j}$, $B_{k_j \ell_j}$, $C_{k_j \ell_j}$, $D_{k_j \ell_j}$; we are free to choose them to satisfy (13). Only in this way can we avoid ξ or η proportional terms in v_1.

Recalling (7), it is clear that the only a (K,L) which contribute to the right hand side of (13) are the a (k_j, ℓ_j). Equating real and imaginary parts of the left and right hand sides of (13),

$$k_j B_{k_j \ell_j}(a) + \ell_j A_{k_j \ell_j}(a) = \mathrm{Im}\left\{F_{k_j \ell_j}(a)\right\}$$

$$a\,(k_j, \ell_j)\,(\ell_j C_{k_j \ell_j}(a) + k_j D_{k_j \ell_j}(a)) = -\mathrm{Re}\left\{F_{k_j \ell_j}(a)\right\}, \qquad (14)$$

all k_j, ℓ_j.

One consequence of (14) and (4) is worth noting right away, in connection with the closure problem: we can never have, simultaneously, $a(k_j, \ell_j) = 0$ and $F_{k_j \ell_j}(a) \neq 0$ for any j. Thus all the various $\varepsilon = 0$ normal modes of the system, if coupled by the $\mathcal{F}$, must be excited to 0 (1). This leaves us with two possibilities: the subset of the basic vectors for which

$$F_{k_j \ell_j}(a) \neq 0$$

$$a\,(k_j, \ell_j) \neq 0$$

is either finite or infinite. Naturally, the former case has many more calculational possibilities. The latter case is calculable when two circumstances happen to exist

$$\mathrm{Im}\left\{F_{k_j \ell_j}(a)\right\} = 0 \qquad \text{all } j;$$

$$a\,(k_j, \ell_j) \neq 0 \qquad , \text{ all } j \text{ for which } F_{k_j \ell_j}(a) \neq 0$$

It is regrettable that more satisfying general statements about when it is possible to achieve closure among the (k_j, ℓ_j) cannot be made. This is due to the very general possibilities for $\mathcal{F}$; to go much farther, we must specialize $\mathcal{F}$, which we do in Sec. IV.

We close this section with a proof that the situation in reference 1, with only one monochromatic wave in zeroth order, always leads trivially to closure. In ref. 1, we had

$$f^{(o)} = a\,(k_o,\ \ell_o)\ e^{\ i\psi(k_o,\ \ell_o)} + a\,(-k_o, -\ell_o)\ e^{\ i\psi(-k_o,\ -\ell_o)},$$

with $k_o\ \ell_o = \Lambda^2$, and all other $a(k_j,\ \ell_j) = 0$. From $\mathcal{F}(f^{(o)},\ \partial f^{(o)}/\partial\xi,\ \partial f^{(o)}/\partial\eta\ )$ we get terms of the type

$$F_{k_j\,\ell_j}\ (a) \neq 0 \qquad \text{only if} \qquad \begin{aligned} k_j &= (n_1 + n_2 + \ldots)\ k_o \\ \ell_j &= (n_1 + n_2 + \ldots)\ \ell_o \end{aligned}$$

If $k_j\ell_j = \Lambda^2$, and $k_o\ell_o = \Lambda^2$, this implies

$$(n_1 + n_2 + \ldots)^2 = 1\,,$$

or $(k_j,\ \ell_j) = \pm(k_o,\ \ell_o)$, so that we can never be led outside the set $(k_o,\ \ell_o)$, $(-k_o,\ -\ell_o)$ by any form of $\mathcal{F}$.

III. THE ROLE OF BOUNDARY CONDITIONS

From (3) and (4), it is clear that $A_{k_j\ell_j}$ (a) and $B_{k_j\ell_j}$ (a) are not completely independent, nor are $C_{k_j\ell_j}$ (a) and $D_{k_j\ell_j}$ (a). Such connection as exists between them is largely determined by boundary conditions, which we now pause to discuss.

In Sec. I, we limited ourselves to the case where f and its normal derivative are given as periodic along the x or t axes. It is clear from the form of Eq. (1) that periodicity, once given, will be preserved throughout the rest of the region of interest in the xt plane in both cases. Inspection of our solution also makes it obvious that it is the interval of periodicity which fixes the spectra of the basic vectors.

From these considerations, it follows that if the boundary is the x axis and the interval of periodicity is L

$$
\begin{aligned}
k_j - \ell_j &= 2\pi jc/L, \\
k_j \ell_j &= \Lambda^2, \\
C_{k_j \ell_j} &= D_{k_j \ell_j}, \\
A_{k_j \ell_j} &= B_{k_j \ell_j}, \quad j = 0, \pm 1, \pm 2, \ldots .
\end{aligned}
\tag{15}
$$

Similarly, if the periodicity is in t with period T,

$$
\begin{aligned}
k_j + \ell_j &= 2\pi jc/L, \\
k_j \ell_j &= \Lambda^2
\end{aligned}
$$

$$C_{k_j \ell_j} = - D_{k_j \ell_j}$$
$$A_{k_j \ell_j} = - B_{k_j \ell_j} \qquad (16)$$
$$j = \pm 1, \pm 2, \ldots$$

In both cases, $k_j + \ell_j$ is the frequency of the jth zeroth order mode, and $(k_j - \ell_j)/c$ is the wave number. In the former case, we call $2\varepsilon C_{k_j \ell_j}$ the jth "frequency shift" and in the latter case, $2\varepsilon C_{k_j \ell_j}/c$ is the jth "wave number shift".

In both cases, the program is, in words, the following. Determine the a (k_j, ℓ_j) and $\psi\ (k_j, \ell_j)$ over the boundary from the boundary conditions. Compute the $F_{k_j \ell_j}$ (a) there, using the a (k_j, ℓ_j) and $\psi(k_j, \ell_j)$. From Eq. (4) and the first of Eqs. (14), we then have a system of differential equations for the a (k_j, ℓ_j) which may or may not have a simple solution. Assuming that this system is solvable for the a (k_j, ℓ_j), we can then solve the second of Eqs. (14) for the frequency or wave number shifts, as the case may be. We then solve Eqs. (3) and (5) for the $\psi(K,L)$. Finally, we determine v_1 from (12), subject to some boundary conditions.

The most convenient set of boundary conditions for v_1 is to choose v_1 and its normal derivative to be zero over the boundary, which is equivalent to putting all of f into the zeroth order over the boundary. This is by no means the only possible split-up, of course.

It need not be said that this program is too ambitious to carry out for all $\mathcal{F}$'s and $f^{(0)}$'s, which is not surprising, since only in special cases can the much simpler program for the harmonic oscillator be pushed to completion[3]. It can be carried out for several interesting cases, and we shall now illustrate two of the simpler of these.

IV. TWO TRACTABLE CASES

A. <u>A Case in which $\mathcal{F} = \mathcal{F}(f)$ only.</u>

In this case, $\mathrm{Im}\left\{F_{k_j \ell_j}(a)\right\} = 0$, all j, so we may pick $A_{k_j \ell_j} = B_{k_j \ell_j} = 0$ for all j, and $a(k_j, \ell_j) = \text{const.}$, all j. This leads at once to

$$\psi(k_j, \ell_j) = (k_j + \varepsilon C_{k_j \ell_j})\xi + (\ell_j + \varepsilon D_{k_j \ell_j})\eta + \phi(k_j, \ell_j), \tag{17}$$

all j, where $\phi(k_j, \ell_j)$ is a constant.

For definiteness, let us give f and its normal derivative over the x axis in the form

$$f(x, o) = \sum_n c_n e^{2\pi inx/L}, \qquad c_n = c^*_{-n},$$

$$\frac{\partial f(x,o)}{\partial t} = \sum_n d_n e^{2\pi inx/L}, \qquad d_n = d^*_{-n}.$$

The function f must be matched up with the t = 0 value of

$$f^{(o)} = \sum_{k_j,\ell_j} a(k_j,\ell_j)\, e^{i\psi(k_j,\ell_j)}$$

$$= \sum_{k_j,\ell_j} a(k_j,\ell_j)\, e^{i(k_j - \ell_j)x/c + i\phi(k_j,\ell_j)}$$

In this "Fourier series", the coefficient of $e^{i(k_n - \Lambda^2/k_n)x/c}$ is $a(k_n, \Lambda^2/k_n)\, e^{i\phi(k_n, \Lambda^2/k_n)} + a(-k_n\, -\Lambda^2/k_n)\, e^{i\phi(-\Lambda^2/k_n, -k_n)}$, so that

$$c_n = a(k_n, \Lambda^2/k_n)\, e^{i\phi(k_n, \Lambda^2/k_n)} + a(-\Lambda^2/k_n, -k_n)\, e^{i\phi(-\Lambda^2/k_n, -k_n)} \qquad (18)$$

The derivative is

$$\frac{\partial f^{(o)}}{\partial t} = \sum_{k_j, j} i(k_j + \ell_j)\, a(k_j,\ell_j)\, e^{i\phi(k_j,\ell_j) + i(k_j - \ell_j)x/c}$$

so that

$$d_n = i\,(k_n + \Lambda^2/k_n)\ \ [a\,(k_n, \Lambda^2/k_n)\, e^{i\phi\,(k_n, \Lambda^2/k_n)}$$
$$- a\,(-\Lambda^2/k_n, -k_n)\, e^{i\phi\,(-\Lambda^2/k_n, -k_n)}]. \tag{19}$$

Solving (18) and (19)

$$a\,(k_n, \Lambda^2/k_n)\, e^{i\phi\,(k_n, \Lambda^2/k_n)}$$
$$= 1/2\ \ [c_n + d_n/i\,(k_n + \Lambda^2/k_n)], \tag{20}$$

which determines $a(k_j, \ell_j)$ and $\phi(k_j, \ell_j)$ for all j. This therefore determines $\psi(k_j, \ell_j)$, all j. We assume throughout the rest of this subsection that we know the a's and ϕ's.

Now we specialize to study the nonlinear effects associated with two zeroth order travelling waves in the presence of an $\mathcal{F} = -\ \nu f^3$, ν = const. (The situation is then equivalent to a stretched string imbedded in a non-linear elastic medium with the end points fixed, if a concrete physical example is desired.) We determine, in particular, the frequency shifts for waves 1 and 2, with

$$f^{(o)} = a_1 \cos\psi_1 + a_2 \cos\psi_2 \tag{21}$$

where for $\varepsilon = 0$

$$\psi_1 = k_1\xi + \ell_1\eta + \phi_1, \qquad \psi_2 = k_2\xi + \ell_2\eta + \phi_2, \tag{22}$$

$$a(k_1, \ell_1) = a(-k_1, -\ell_1) = a_1/2, \qquad a(k_2, \ell_2) = a(-k_2, -\ell_2) = a_2/2.$$

Closure can be guaranteed by picking $(k_1/k_2) + (k_2/k_1)$ as any irrational number, though weaker conditions will suffice.

Computing $\mathcal{F}(f^{(o)})$, it is readily seen that the non-vanishing $F_{KL}(a)$'s are:

(K,L)	$-8F_{KL}(a)/\nu$
$3(k_1, \ell_1)$	a_1^3
(k_1, ℓ_1)	$3a_1^3 + 6a_1a_2^2$
$3(k_2, \ell_2)$	a_2^3
(k_2, ℓ_2)	$3a_2^3 + 6a_1^2a_2$
$(k_1, \ell_1) + 2(k_2, \ell_2)$	$3a_1a_2^2$
$(k_1, \ell_1) - 2(k_2, \ell_2)$	$3a_1a_2^2$
$2(k_1, \ell_1) + (k_2, \ell_2)$	$3a_1^2a_2$
$2(k_1, \ell_1) - (k_2, \ell_2)$	$3a_1^2a_2$

with $F_{KL}(a) = F_{-K,\,-L}(a)$, all (K,L). Eqs. (14) and (15) now give:

$$\frac{a_1}{2}\ C_{k_1 \ell_1}(a)\ [\Lambda^2/k_1 + k_1] = (\nu/8)\ (3a_1^3 + 6\ a_1 a_2^2)$$

$$\frac{a_2}{2}\ C_{k_2 \ell_2}(a)\ [\Lambda^2/k_2 + k_2] = (\nu/8)\ (3a_2^3 + 6\ a_1^2\ a_2),$$

which gives the frequency shifts for waves 1 and 2:

$$\begin{aligned} \Delta\omega_1 &\equiv 2\epsilon C_{k_1\ell_1} = \frac{3\epsilon\nu}{2}\ (a_1^2 + 2a_2^2)\ /\ (k_1 + \Lambda^2/k_1) \\ \Delta\omega_2 &\equiv 2\epsilon C_{k_2\ell_2} = \frac{3\epsilon\nu}{2}\ (a_2^2 + 2a_1^2)\ /\ (k_2 + \Lambda^2/k_2). \end{aligned} \tag{23}$$

The computations for a(K,L) for (K,L) outside the basic set of vectors and for v_1 are completely straightforward, and will not be written out.

B. A Dissipative Case

We consider again the travelling waves of Eqs. (21) and (22), but now with a frictional dissipation, $F = -4\sigma(\partial f/\partial t)^3$. (The case of linear friction, $F \sim -\partial f/\partial t$, is trivial.) Thus

$$\mathfrak{F}(f^{(o)}) = -\sigma\left(\frac{\partial f^{(o)}}{\partial \xi} + \frac{\partial f^{(o)}}{\partial \eta}\right)^3 \qquad (24)$$

$$= \sigma\,[\,(k_1 + \ell_1)a_1 \sin\psi_1 + (k_2 + \ell_2)\,a_2 \sin\psi_2\,]^3$$

The F_{KL} (a) are again readily computed, and all have zero real parts, so that the shifts all vanish. The nonvanishing $F_{KL}(a)$ are:

(K,L)	$-8iF_{KL}(a)/\sigma$
$3(k_1,\ell_1)$	α_1^3
(k_1,ℓ_1)	$-(3\alpha_1^3 + 6\alpha_1\alpha_2^2)$
$3(k_2,\ell_2)$	α_2^3
(k_2,ℓ_2)	$-(3\alpha_2^3 + 6\alpha_1^2\alpha_2)$
$(k_1,\ell_1) + 2(k_2,\ell_2)$	$3\alpha_1\alpha_2^2$
$(k_1,\ell_1) - 2(k_2,\ell_2)$	$3\alpha_1\alpha_2^2$
$2(k_1,\ell_1) + (k_2,\ell_2)$	$3\alpha_1^2\alpha_2$
$2(k_1,\ell_1) - (k_2,\ell_2)$	$-3\alpha_1^2\alpha_2$

where $\alpha_1 = a_1 (k_1 + \ell_1)$, $\alpha_2 = a_2 (k_2 + \ell_2)$, and $F_{-K,-L}(a) = - F_{KL}(a)$.

We find, from (14),

$$(k_1 + \Lambda^2/k_1)\, A_{k_1\ell_1}(a) = -\sigma/8\,(3\alpha_1^3 + 6\,\alpha_1\,\alpha_2^2)$$
$$(k_2 + \Lambda^2/k_2)\, A_{k_2\ell_2}(a) = -\sigma/8\,(3\alpha_2^3 + 6\,\alpha_2\,\alpha_1^2). \tag{25}$$

Taking into account (15), we have a_1 and a_2 developing according to

$$\frac{\partial a_1}{\partial t} = 2\varepsilon A_{k_1\ell_1}(a)$$

$$= \frac{-\sigma\varepsilon}{4(k_1 + \Lambda^2/k_1)}\,[3\,(k_1 + \ell_1)^3 a_1^3 + 6\,(k_1 + \ell_1)\,(k_2 + \ell_2)^2 a_1 a_2^2\,]$$

$$\frac{\partial a_2}{\partial t} = 2\,\varepsilon A_{k_2\,\ell_2}(a) \tag{26}$$

$$= -\frac{\sigma\varepsilon}{4\,(k_2 + \Lambda^2/k_2)}\,[3\,(k_2 + \ell_2)^3\,a_2^3 + 6(k_2 + \ell_2)\,(k_1 + \ell_1)^2\,a_1^2\,a_2].$$

Eqs. (26) determine the damping of the oscillations in time as t increases from zero. Since the $\psi(k_j, \ell_j)$ retain their simple $k_j\xi + \ell_j\eta$ + const. form for all x,t, the rest of the solution is straightforward. It is easily shown from (26) that the damping of a_1 and a_2 is $O(1/\sqrt{\varepsilon t})$.

ACKNOWLEDGMENTS

Valuable discussions with Drs. D.A. Tidman, J.K. Hale, and A. Stokes are gratefully acknowledged.

FOOTNOTES

1. D. Montgomery and D.A. Tidman, Phys. Fluids <u>7</u>, 242 (1964).

2. An elegant approach to the wave equation using similar methods has been given by M.D. Kruskal and N.J. Zabusky, in J. Math. Phys. <u>5</u>, 231 (1964). The perturbation series for the wave equation, however, possesses much more pathological behavior than in the case of the Klein-Gordon equation.

3. N. Bogolyubov and Y.A. Mitropolskii, <u>Asymptotic Methods in the Theory of Nonlinear Oscillations</u> (Gordon and Breach Science Publishers, New York, 1961; translated from the Russian).

4. See, e.g., P.M. Morse and H. Feshbach, <u>Methods of Theoretical Physics,</u> Vol. I, Chapter 6, (McGraw Hill Book Co., New York, 1953).

5. A quite similar recipe to that given on p. 686 of Reference 4 for the wave equation can be readily given for the Klein-Gordon equation.